Topological Group

The second course

الدراسات العليا / ماجستير

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3.2. The Topological Space βD

In this section we define a topology on the set of all ultrafilters on a set D.

Def (3.15): Let D be a discrete topological space

(a) $\beta D = \{ p ; p \text{ is an ultrafilter on } D \}$ (filter)

(b) Given $A \subseteq D$, $\hat{A} = \{ p \in \beta D ; A \in p \}$

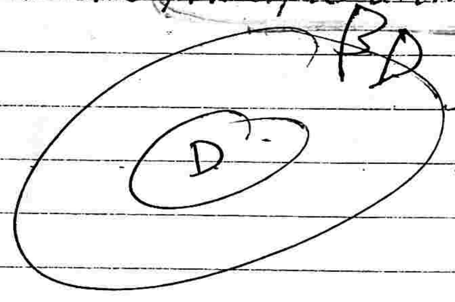
Note: we shall be thinking of ultrafilters as points in a topological space.

Def 3.16: Let D be a set and let $a \in D$. Then

$e(a) = \{ A \subseteq D ; a \in A \}$ لاحظ إنه $e(a)$ هو عنصر من عناصر $\hat{A}$ أي ينتمي إلى $\hat{A}$

element a

* Hence for each $a \in D$, $e(a)$ is the principle ultrafilter corresponding to a



Lemma 3.17: Let D be a set and let $A, B \subseteq D$

(a) $\widehat{(A \cap B)} = \hat{A} \cap \hat{B}$

Proof: Let $p \in \widehat{A \cap B}$, p is Altrafilter and $\underset{\subseteq D}{A \cap B} \in p$

$\Rightarrow A \in p$, $B \in p$

by def $p \in \hat{A}$ and $p \in \hat{B}$ {from def of ultrafilter {Since $A \cap B \subseteq A \subseteq D \Rightarrow A \in p$} similarly for B}

Hence $\widehat{A \cap B} \subseteq \hat{A} \cap \hat{B}$ ——(1) $\hat{A} = \{p \in \beta D : A \in p\}$

by def

Now, let $p \in \hat{A} \cap \hat{B}$ where p is a filter, $A \in p$, $B \in p$

Since $A \in p$ and $B \in p$

$\Rightarrow$ by def of filter $A \cap B \in p$ فلتر

$\Rightarrow$ by def of filter $A \cap B \in p$] filter من شروط الفلتر $\cap \in$ زائدة

$\Rightarrow p \in \widehat{A \cap B}$

(b) $(\widehat{A \cup B}) = \hat{A} \cup \hat{B}$

Sol: Let $p \in \widehat{A \cup B}$ where p is an ultrafilter] or by def $\hat{A}$

$\Rightarrow A \cup B \in p$

Suppose $A \notin p$, $B \notin p$

$\Rightarrow A^c, B^c \in p$

$\Rightarrow A^c \cap B^c \in p$ since p ~~is an~~ DeMorgan ultrafilter.

$\Rightarrow (A \cup B) \cap (A^c \cap B^c) = (A \cup B) \cap (A \cup B)^c = \phi \in p$ C!

Hence $\widehat{A \cup B} \subseteq \hat{A} \cup \hat{B}$ —(1) $\Rightarrow A \in p, B \in p$ $\Rightarrow p \in \hat{A}, p \in \hat{B}$ (b) def $\Rightarrow p \in \hat{A} \cup \hat{B}$

$\Leftarrow$ let $p \in \hat{A} \cup \hat{B}$ $\in$ by def. $\in$

$\Rightarrow p \in \hat{A}$ or $p \in \hat{B}$

if $p \in \hat{A}$ {by def $\hat{A}$

$\Rightarrow A \in p$

since $A \subseteq A \cup B \subseteq D$

$\Rightarrow$ by def of ultrafilter $A \cup B \in p$

$\Rightarrow p \in \widehat{A \cup B}$

similarly, if $p \in \hat{B} \rightarrow B \in p$, since $B \subseteq A \cup B \subseteq D$

$\Rightarrow A \cup B \in p \rightarrow p \in \widehat{A \cup B}$.

(c) $(\widehat{D \setminus A}) = \beta D \setminus \hat{A}$

Sol: $(\widehat{D \setminus A}) = \{p \in \beta D : A^c \in p\}$

So let $p \in \beta D$ where $A^c \in p$ by def

$\Rightarrow A \notin p$ {since p is ultrafilter}

this mean $p \notin \hat{A}$

$\Rightarrow p \in \beta D \setminus \hat{A}$

$\Leftarrow$ $\beta D \setminus \hat{A} = \{p \in \beta D : A \notin p\}$

let $p \in \beta D \setminus \hat{A}$

$\Rightarrow A \notin p$

$\Rightarrow A^c \in p$

$\Rightarrow p \in D \setminus \hat{A}$

(d) $\hat{A} = \phi$ iff $A = \phi$

Sol: Assume $A \neq \phi$

$\Rightarrow \exists\, a \in A$

$\Rightarrow e(a)$ is p. Alt. f

$\Rightarrow A \in (a)$ by def of p. Alt f

$\Rightarrow \hat{A} \neq \phi$ ✗

$\Leftarrow$ suppose $A = \phi$

since $\phi \notin p$ for any ultrafilter

$\Rightarrow \hat{A} = \phi$

(e) $\hat{A} = \beta D$ iff $A = D$ H.W

(f) $\hat{A} = \hat{B}$ iff $A = B$

Sol: suppose $\hat{A} = \hat{B} = \{p \in \beta D ; A, B \in p\}$

suppose $A \neq B$

$\Rightarrow \exists\, x \in A$ & $x \notin B$

note $e(x) = \{A \subseteq D ; x \in A\}$ p. Alt. f

$\Rightarrow$ by def $A \in e(x)$, $B \notin e(x)$

So $e(x) \in \hat{A}$ & $e(x) \notin \hat{B}$ ✗

$\Leftarrow$ suppose $A = B$

by def $\hat{A} = \{p \in \beta D : A \in p\}$

$= \{p \in \beta D : B \in p\}$ {since $A = B$}

$= \hat{B}$

let (X,T) be a top. space, a sub collection β of T ... iff for every member of T is the union of members of B

Note! From lemma 3.17, the set $\{\hat{A}\}$ are closed under finite intersection, because $\hat{A} \cap \hat{B} = \widehat{A \cap B}$

$\Rightarrow \{\hat{A} : A \subseteq D\}$ form a basis for topology on βD

because ~~{if A has the property that all finite intersections of elt of A are also unions of elt. of A~~

and we we will define the topology of βD to be the topology ~~of βD to~~ which has these sets as a basis.

Thm 3.18: Let D be any set

(a) βD is a compact T_2 - space

Proof! Suppose that p and q (ultrafilter) be two distinct elt of βD

بالطبع لأنه قلنا إنه الآن التبولوجي على D التبولوجي الbase

if $A \in p \setminus q$ then $\hat{A}^c = D \setminus A \in q$

So $\hat{A}$ and $\widehat{D \setminus A}$ are disjoint open subsets of βD containing p & q resp,

Hence βD is T_2 - space $\boxed{\cdot}$

To show its compact

a base for a closed sets of top. space X is a family of closed sets F s.t any closed set A is intersection of members of F.

the set ~~of~~ the form $\hat{A}$ are also a base for the closed sets

because $\underbrace{\beta D \setminus \hat{A}}_{\text{closed}} = \widehat{D \setminus A}$ ← open since its belong to family

A space X is compact iff every collection of closed subsets of X satisfy the intersection property has non empty intersection. F.I.P

To show βD is compact, we will show that a family $\mathcal{A} = \{$the set of the form $\hat{A}$ with finite intersection property$\}$ has non-empty intersection.

let $\mathcal{B} = \{A \subseteq D ; \hat{A} \in \mathcal{A}\}$

if $\mathcal{F} \in \mathcal{P}_f(\mathcal{B}) = \{\mathcal{F} ; \phi \neq \mathcal{F} \subseteq \mathcal{B}$ and $\mathcal{F}$ is finite$\}$

this mean for each $A_i \in \mathcal{F}$, $\hat{A}_i \in \mathcal{A}$

from def for $\mathcal{A}$ above there is some $p \in \bigcap_{A \in \mathcal{F}} \hat{A}$

and so $\bigcap \mathcal{F} \in p$ $\{$from def of $\hat{A} \Rightarrow \exists\, p$ s.t $A \in p$

and thus $\bigcap \mathcal{F} \neq \phi$

Hence $\mathcal{B}$ has f.I.P

So by Thm 3.8 $\exists$ an ultra filter $q \in \beta D$ s.t $\mathcal{B} \subseteq q$

$\Rightarrow q \in \bigcap \mathcal{A}$

So every collection of closed subsets of X satisfy the f.I.p has non empty intersection

$\Rightarrow \beta D$ is compact.

(b) The set of the form $\hat{A}$ are the clopen subsets of βD.

Sol: As we said in part (a) the set $\hat{A}$ is clopen subsets of βD

الآن نريد أن نثبت أي مجموعة open و closed في فضاء العائلة

So, suppose that C is any clopen subset of βD

since Base → في C

let $\mathcal{A} = \{ \hat{A} ; A \subseteq D \text{ and } \hat{A} \subseteq C \}$

since C is open

$\Rightarrow \mathcal{A}$ is open cover of C {since we say $\hat{A}$ is basis}

and since C is closed

لأن C مغلقة

$\Rightarrow C$ is compact {since from (a) we say βD is compact T_2-space $\Rightarrow$ we know any closed subset of compact is compact}

حسب التعريف يمكن إيجاد غطاء منتهي

So by def of compact we can pick a finite subfamily $\mathcal{F}$ of $p(D)$ s.t $C = \bigcup_{A \in \mathcal{F}} \hat{A}$ {since by def of compact $C \subseteq \cup \hat{A}$ and from def of $\mathcal{A}$ each $\hat{A} \subseteq C$ $\Rightarrow$ we have $=$}

لا يلزم

$\Rightarrow$ by lemma 3.17 (b), $C = \widehat{\bigcup \mathcal{F}}$ يعني تم تغطيتها بشكل ^

= is the set of all principle ultra. f. generated by elt from A
= $\{e(a) ; a \in A\}$

(c) For every $A \subseteq D$, T.P $\hat{A} = cl_{\beta D}\, e[A]$ closure

Soll $\Rightarrow$ let $a \in A$

$\Rightarrow$ $e(a) \in \hat{A}$ by def of $e[A]$

$\Rightarrow e[A] \subseteq \hat{A}$

$\Rightarrow \overline{e[A]} \subseteq \hat{A}$ $\leftarrow$ since $\hat{A}$ closed

Hence $cl_{\beta D}\, e[A] \subseteq \hat{A}$

alt-g $\to$ by def

$\Leftarrow$ Let $p \in \hat{A} = \{p \in \beta D ; A \in p\}$

If $\hat{B} = \{p \in \beta D ; B \in p\}$ denoted a basic neigh of p

اعتقد الجوار سيكون هو مجموعة $\hat{B}$ التي تحوي على مجموعة مفتوحة (والتي بهذه الحالة سيكون $\hat{B}$ نفسه لانه قلنا إنه المجموعة المفتوحة في βD هي بهذا الشكل) والتي تحوي p

لانها

So $B \in p$ & $A \in p$

$\Rightarrow A \cap B \in p$

$\Rightarrow A \cap B \neq \phi$

let $a \in A \cap B$

since $e(a) = \{A \subseteq D ; a \in A\} \in e[A] \cap \hat{B}$

So $e[A] \cap \hat{B} \neq \phi$

Hence $p \in cl_{\beta D}\, e[A]$

d) For any $A \subseteq D$ and any $p \in \beta D$, $p \in cl_{\beta D}\, e[A]$ iff $A \in p$

Proof) let $p \in cl_{\beta D}\, e[A] \overset{\text{by } c}{\Longleftrightarrow} p \in \hat{A} \overset{\text{by def of } \hat{A}}{\Longleftrightarrow} A \in p$

idea Plus

(e) The mapping e is injective and $e[D]$ is a dense subset of βD whose points are precisely the isolated points of βD

Def/ A subset A of Top. space X is dense in X if every point $x \in X$ is either belongs to A or is a limit point of A. $(\bar{A} = A)$

Def/ a point x in top space X is called isolated point of a subset S of X if its belong to S and $\exists$ a neigh of x not containing other points of S.

proof/ let $a \neq b \in D$

then $\{a\}^c = D \setminus \{a\} \in e(b) \setminus e(a)$

$\left\{ \begin{array}{l} \text{since by def } e(a) = \{A \subseteq D ; a \in A\} \\ \text{which is an ult. f generated by} \\ a \text{ and } \{a\} \in e(a) \text{ since } a \in \{a\} \\ \text{and since } b \notin \{a\} \Rightarrow \{a\} \notin e(b) \\ \text{but } e(b) \text{ its ult. f} \Rightarrow \{a\}^c \in e(b) \end{array} \right.$

Hence $e(a) \neq e(b)$

so e is 1-1

To show $e[D]$ is dense subset of βD

we need to show $e[D]$ has its point and a limit point of $e[D]$.

So we will try to show if p (ult.f) is a point in βD is a limit point of $e[D]$ if every neig of p contains at least one point of $e[D]$ diff from p itself. تعريف نهائي

يعني سنحاول في الـ
انه لكل نقطة و
لهذه النقط
بحيث إن تقاطع
هذا الجوار مع
$\phi \neq e[D]$

هذا هو الجوار

So, let $\hat{A}$ is a non basic open subset of βD

Then $A \neq \phi$ $\{$by def of $\hat{A} = \{p \in \beta D ; A \in p\}$

لانه $e(a)$ عنصر في $\hat{A}$

any $a \in A$ satisfy $e(a) \in e[D] \cap \hat{A}$

حسب التعريف

and so $e[D] \cap \hat{A} \neq \phi$

i.e $e[D]$ has its limit point

Thus $e[D]$ is dense in βD

Now to show the points of βD are isolated

note that for any $a \in D$, $e(a)$ $\overset{e[D]}{\in}$ is isolated in βD because $\{e(a)\} = \{\hat{a}\}$ is an open subset of βD whose only member is $e(a)$

So by def of isolated point $e(a)$ is an isolated point.

i.e βD is not discreat

Conversly, if p is an isolated point of βD i.e $\{p\}$ is open

then $\{p\} \cap e[D] \neq \phi$ {because $e[D]$ is dense so any open set $\cap e[D] \neq \phi$}

Hence $p \in e[D]$.

(f) if $\mathcal{U}$ is an open subset of βD, $cl_{\beta D}\,\mathcal{U}$ is also open.

Proof: if $\mathcal{U} = \phi$. is trival

* $e^{-1}[\mathcal{U}]$ the sets in D it maps to $\mathcal{U}$

if $\mathcal{U} \neq \phi$

let put $A = e^{-1}[\mathcal{U}] \neq \phi$ $\left\{ \begin{array}{l} \text{since } e[D] \text{ is dense in } \beta D \\ \Rightarrow \mathcal{U} \cap e[D] \neq \phi \end{array} \right.$

we claim $\mathcal{U} \subseteq cl_{\beta D}\, e[A]$

let $p \in \mathcal{U}$ and let $\hat{B} = \{p' \in \beta D ; B \in p'\}$ be a basic neigh of p (open set)

then $\mathcal{U} \cap \hat{B}$ is non empty open set $\left\{ \begin{array}{l} \text{since } \mathcal{U} \text{ \& } \hat{B} \text{ are open} \\ \text{and since } p \in \mathcal{U} \text{ \& } p \in \hat{B} \end{array} \right.$

from above $e[D]$ dense so intersection with any open set $\neq \phi$

$\Rightarrow$ by part (e), $\mathcal{U} \cap \hat{B} \cap e[D] \neq \phi$

أخذنا هنا الكل بنهاية في e[D]
وهنا كل عناصره

So let $b \in B$ with $e(b) \in \mathcal{U}$

then $e(b) \in \overset{\text{by def}}{\hat{B}} \cap e[A]$ $\left\{ \begin{array}{l} \text{since } e[A] = e[e^{-1}(\mathcal{U})] \subseteq \mathcal{U} \\ \text{and we suppose } e(b) \in \mathcal{U} \end{array} \right\}$

Hence $\hat{B} \cap e[A] \neq \phi$

جوار ، أكبر $\subseteq \mathcal{U}$

* and so $\hat{B} \cap e[A] \neq \phi$

$\Rightarrow p \in cl\, e[A]$ حققنا التعريف

Also by def of A, $e[A] \subseteq \mathcal{U}$

Hence $\mathcal{U} \subseteq cl_{\beta D}\, e[A] \subseteq cl_{\beta D}\, \mathcal{U}$

Hence $cl_{\beta D} \mathcal{U} = cl_{\beta D}\, e[A] = \hat{A}$ (by c) --- (*)

since from above we say $\mathcal{U} \subseteq cl_{\beta D}\, e[A]$

$\Rightarrow \bar{\mathcal{U}} \subseteq cl_{\beta D}\, e[A]$

and since we have $e[A] \subseteq \mathcal{U} \Rightarrow \overline{e[A]} \subseteq \bar{\mathcal{U}}$

Hence we get the $=$.

or from (*) since the closure is the smallest closed set contain set

so $cl_{\beta D} \mathcal{U}$ is the smallest closed set containing $\mathcal{U}$

and this mean there is no any closed between them and

hence $cl_{\beta D} \mathcal{U} = cl_{\beta D}\, e[A]$

Hence $cl_{\beta D} \mathcal{U} \overset{=\hat{A}}{}$ is open since $\hat{A}$ is clopen.

~~we will establish a characterization of the closed subsets of βD~~

Defl Let D be a set and let A be a filter on D.

then $\hat{A} = \{ p \in \beta D ; A \subset p \}$ يعني كل ال ult.f اي تحوي هذا الفلتر

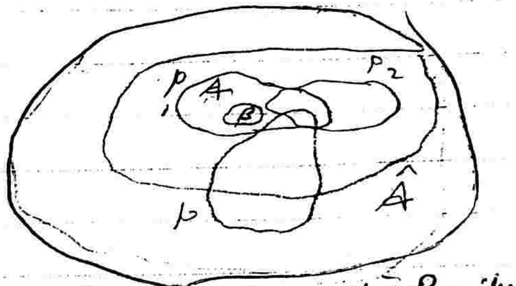
د. ليث

Thm 3.20: Let D be a set

(a) If $\mathcal{A}$ is a filter on D, then $\hat{\mathcal{A}}$ is a closed subset of βD.

$\{p \in \beta D : \mathcal{A} \subseteq p\}$

proof let $p \in \beta D \setminus \hat{\mathcal{A}}$ $\{$i.e p is ult. f not contain $\mathcal{A}$ [illegible] $\hat{\mathcal{A}}\}$



pick $B \in \mathcal{A} \setminus p$

يعني [illegible] Point then $\widehat{D \setminus B} = \{q \in \beta D : B^c \in q\}$ is a neigh of p $\{$since the family of $\wedge$ is a base$\}$

which not contain $\hat{\mathcal{A}}$

i.e $\hat{\mathcal{A}}^c$ is open

$\Rightarrow \hat{\mathcal{A}}$ is closed

(b) If $\phi \neq A \subseteq \beta D$ and $\mathcal{A} = \cap A$, then $\mathcal{A}$ is a filter on D and $\hat{\mathcal{A}} = cl\, A$

H.W

Sol. since $\mathcal{A}$ is the intersection of set of filters

$\Rightarrow \mathcal{A}$ is a filter $\}$

and as we said A is the set of filters

So for each $p \in A \Rightarrow \mathcal{A} \subseteq p$ $\{$since $\mathcal{A} = \cap A\}$

filter [illegible]

$\Rightarrow A \subseteq \hat{\mathcal{A}}$ $\{$ by def of $\hat{\mathcal{A}}\}$

$\Rightarrow cl(A) \subseteq \hat{\mathcal{A}}$ $\{$by part (a) $\hat{\mathcal{A}} \subseteq cl\, A\}$ —— (1)

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To show $\hat{A} \subseteq cl A$

let $\overset{\text{ult f}}{p} \in \hat{A} = \{p \in BD : A \in p\}$

& let $B \in p$, $\hat{B} = \{q \in BD : B \in q\}$

open set contain B

Suppose $\hat{B} \cap A = \phi$

then for each $\overset{\text{ult f}}{q} \in A$, $D \setminus B \in q$ ← Since each elt f contain either A or A^c

So $D \setminus B \in A = p$ {from def of A} C! Since p contain B and B^c

P.24 Def $A^* = \hat{A} \setminus A$

Def: Let D be a set and let $A \subseteq D$. Then $\hat{A}^* = \hat{A} \setminus e[A]$

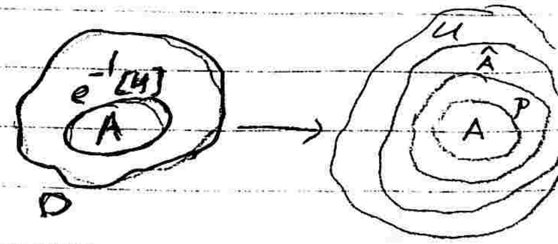
the set of all principle ult f generated by elt from $A = \{e(a) ; a \in A\}$.

Thm 3.22: Let $p \in \beta D$ and let U be a subset of βD. If
once $e(x)$ identify with x the conclusion becomes $U \cap D \in p$ $\quad e: D \to \beta D$
U is a neigh. of p in βD, then $e^{-1}[U] \in p$ $\quad e^{-1}[U]$ is the set in D it maps to U

Proof: let U be a neigh of p

So, $\exists$ a basic open subset $\hat{A}$ of βD s.t. $p \in \hat{A} \subseteq U$

$\Longrightarrow$ by def of $\hat{A}$ then $A \in p$



$\leftarrow$ from def of $e^{-1}[U]$
because $A \subseteq e^{-1}[U]$ and $A \in p$

then by (ii) cond of filter $e^{-1}[U] \in p$

3 Stone-Čech compactification

In this section we show that βD is the Stone-Čech compactification of the discrete space D.

* Recall that by an embedding of a topological sypace X into a topological space Z, one means ~~that~~ a function $\varphi : X \to Z$ which defines a homeomorphism from X onto $\varphi[X]$

* Remember we are assuming that all hypothesized topological spaces are Hausdorff

Completly regular space: given any point $x \in X$ and closed subset $A \subseteq X$ such that $x \notin A$ $\exists$ a cont map $f : X \to [0,1]$ s.t $f(x) = 0$ and $f(a) = 1$ $\forall a \in A$

Def Let X be a top. space. A compactification of X is a pair (φ, C) s.t C is a compact space, φ is an embedding of X into C, and $\varphi[X]$ is dense in C

Def: Let X be a completly regular top space. A stone-Čech compactification of X is a pair (φ, Z) such that

(a) Z is a compact space.

(b) φ is an embedding of X into Z,

(c) $\varphi[Z]$ is dense in Z, and

(d) given any compact space Y and any cont. function $f: X \to Y$ there exists a continuous function $g: Z \to Y$ such that $g \circ \varphi = f$. (that is the diagram is comm)

$$\begin{tikzcd} & Z \arrow[dr, "g"] & \\ X \arrow[ur, "\varphi"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

Remark: (a) Any two Stone-Čech compactification of the same topological space X are homeomorphism

i.e if (φ, Z) and (τ, W) be stone čech compactification of X. Then there is a homeomorphism $\gamma: Z \to W$ such that $\gamma \circ \varphi = \tau$

(b) The topology induced on X as a subset of Z is the ~~org~~ original topology of X.

Thm 3.27: Let D be a discrete space. Then $(e, \beta D)$ is a Stone-Čech compactification of D.

proof: Condition (a), (b) and (c) of def 3.25 is hold by Thm 3.18

$$\begin{tikzcd} & \beta D \arrow[dr, "g"] & \\ D \arrow[ur, "e"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

proof: cond (b) and (c)

b) To show e is an embedding

1) To show $e: D \longrightarrow \beta D$ is injective

let $a \neq b \in D$

then $\{a\}^c = D \setminus \{a\} \in e(b) \setminus e(a)$

Hence $e(a) \neq e(b)$

So e is 1-1

2) e is cont since D is discreat space

3) e is closed map

proof: Let $A \subseteq D$ be closed subset

then $e[A] \cap \hat{A} = e[A]$ $\left\{ \begin{array}{l} \text{I think } e[A] \text{ is closed since} \\ \hat{A} = \overline{e[A]} \subseteq e[A]^{\text{[illegible]}} \\ \text{by Thm 3.18} \end{array} \right.$

Since $e[A] \subseteq e[A] \cap \hat{A}$ since let $r \in e[A] \Rightarrow r = (a^{\cdot}) = e(a^{\cdot}),\ a \in A$ $\Rightarrow A \in r \Rightarrow r \in \hat{A}$

$\{B \subseteq D ; a \in B\}$

and clear $e[A] \cap \hat{A} \subseteq e[A]$

subset A of top space X is dense in X if every point $x \in X$ is either [be]longs to A or is a limit point of A ($\bar{A} = A$)

To show $e[D]$ is a dense

we need to show $e[D]$ has its points and a limit point of $e[D]$

So we will try to show if p (ultr. filter) is a point in βD is a limit point of $\beta[D]$ if every neigh of p contains at least one point of $e[D]$ different from p itself

let $\hat{A}$ is a non basic open subset of βD neigh of p

then $\hat{A} \neq \phi$ {by def of $\hat{A} = \{p \in \beta D ; A \in p\}$

$\hat{A}$ ينتمي $e(a)$ بين

any $a \in A$ satisfy $e(a) \in e[D] \cap \hat{A}$

من التعريف

So $e[D] \cap \hat{A} \neq \phi$

i.e $e[D]$ has its limit point

Thus $e[D]$ is dense in βD

d) Given a compact space Y and let $f: D \longrightarrow Y$ be cont

To show 1) there exist 2) a cont fun. $g: \beta D \longrightarrow Y$ 3) s.t its comm

proof! 1) For each $p \in \beta D$ let $A_p = \{cl_Y f[A] ; A \in p\}$.

Then for each $p \in \beta D$, (by Exc 3.3.1), A_p has finite. I. p

$$\begin{tikzcd} & \beta D \arrow[dr, "g"] & \\ D \arrow[ur, "e"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

and since Y is compact

حسب نظرية So A has non-empty intersection

عرفنا الـ → choose $g(p) \in \cap A_p$

idea

2) To show the diagram is comm

let $x \in D$ then $\{x\} \in e(x) = \{A \subseteq D : x \in A\}$

So $g(e(x)) \in cl_Y f[\{x\}] = cl_Y [\{f(x)\}]$
$= \{f(x)\}$ since singlton is closed

So $g \circ f = f$

3) To show g is cont

let $p \in \beta D$ and let U be a neigh of $g(p)$ in Y

Since Y is T_2-compact space then Y is normal space

$\Rightarrow Y$ is regular

So pick a neigh V of $g(p)$ with $cl_Y V \subseteq U$ $\{$ ~~by def of regular~~ we get a closed set

let $A = f^{-1}[V]$ in D

~~claim $A \in p$~~

Suppose $D \setminus A \in p$

$\Rightarrow g(p) \in cl_Y f[D \setminus A]$, ~~and V is a neigh of $g(p)$~~

So $V \cap f[D \setminus A] \neq \phi$ $\boxtimes$ since $A = f^{-1}[V]$

~~Hence $A \in p \Rightarrow p \in \hat{A}$ is a neigh of p~~

claim $g[\hat{A}] \subseteq U$

let $q \in \hat{A} = cl(A)$ and suppose $g(q) \notin U$

open since

$\Rightarrow Y \setminus cl_Y V$ is a neigh of $g(p)$

~~and $g(q) \in cl_Y f[A]$~~

2) To show the diagram is comm

let $x \in D$ then $\{x\} \in e(x) = \{A \subseteq D : x \in A\}$

So $g(e(x)) \in cl_Y f[\{x\}] = cl_Y[\{f(x)\}]$

$= \{f(x)\}$ since singlton is closed

So $g \circ e = f$

3) To show g is cont

let $p \in \beta D$ and let U be a neigh of $g(p)$ in Y

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$\Rightarrow Y$ is regular

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let $q \in \hat{A} = cl(A)$ and suppose $g(q) \notin U$

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$\Rightarrow Y \setminus cl_Y V$ is a neigh of $g(p)$

and $g(q) \in cl_Y f[A]$ $\left\{ \begin{array}{l} \text{since } q \in \hat{A} = cl(A) \Rightarrow f(q) \in f(\bar{A}) \subseteq \overline{f(A)} \\ \text{or since } A \in q \text{ then by directly by def of } g(q) \end{array} \right.$

So $(Y)\, cl_Y V \cap f[A] \neq \phi$ $\{ \text{since } A = f^{-1}[V] \}$

Thm 3.28 (Stone–Čech compactification): Let D be an infinite discrete space. Then

1) βD is a compact space

Proof: by using the Thm {A space X is compact iff every collection of closed subsets of X satisfy the finite intersection property has non empty intersection}:

So we will show a family $\mathcal{A}$ = the sets of the form $\hat{A}$ with the f.I.p has non empty intersection

let $\mathcal{B} = \{ A \subseteq D : \hat{A} \in \mathcal{A} \}$ تعريف اعلاه

If $F \in \mathcal{P}_f(\mathcal{B}) = \{ F : \phi \neq F \subseteq \mathcal{B} \text{ and } F \text{ is finite} \}$

This mean for each $A_i \in F$, $\hat{A}_i \in \mathcal{A}$ {from def of $\mathcal{B}$}

so from def of $\mathcal{A}$ above

there is some ultrafilter $p \in \bigcap_{A \in F} \hat{A}$

and so $\bigcap F \in p$ $\quad \{ p \in \bigcap \hat{A}_i = \bigcap \bar{A}_i$

$\Rightarrow A_i \in p \ \forall i$

$\Rightarrow \phi \neq \bigcap A_i \in p$ since we have finite of them

$\Rightarrow \bigcap F = \bigcap A_i \in p$

and thus $\bigcap F \neq \phi$ $\{ \text{since } \phi \notin p \}$

Hence $\mathcal{B}$ has f.I.p

and $g(q) \in cl_Y f[A]$ $\begin{cases} \text{since } q \in \hat{A} = cl(A) \Rightarrow f(q) \in f(\bar{A}) \subseteq \overline{f(A)} \\ \text{or since } A \in q \text{ then by directly by def of } g(q) \end{cases}$

So $g(y) \in cl_Y(V) \cap f[A] \neq \emptyset$ ⊠ $\{\text{since } A = f^{-1}[V]\}$

Thm 3.28 (Stone – Čech compactification): Let D be an infinite discreate space. Then

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$\Rightarrow \bigcap F = \bigcap A_i \in p$

and thus $\bigcap F \neq \emptyset$ {since $\emptyset \notin p$}

Hence $\mathcal{B}$ has f.I.P

a point x in top. space X is called isolated point of a subset of X if it belongs to S and $\exists$ a neigh of x not containing other points of S.

by Thm 3.8 {Let D be a set and let A be a subset of $P(D)$ which has the f.I.P. then there is an ultrafilter U on D s.t $A \subseteq U$}

in our case βD, B, $P(\beta D)$

So $\exists$ an ultrafilter $q \in \beta D$ s.t $B \subseteq q$

$\Rightarrow q \in \cap A$ $\begin{cases} \Rightarrow \cancel{\text{since } A_i \in B \subseteq q} \\ \Rightarrow A_i \in q \\ \Rightarrow q \in \hat{A}_i \in A \quad \forall i \end{cases}$

b) $D \subseteq \beta D$

Sol. Since we identify the points of D with the principle ultrafilters generated by those points

i.e we identify $s \in D$ with $e(s) \in \beta D$

Hence $D \subseteq \beta D$

(c) D is dense in βD

proof: To show $e[D] = D$ is dense subset of βD

we will try to show if p is ultra. filter in βD is a limit point of D if every neigh of p contains at least one point of D different from p itself.

let U be a neigh of p, so there is a basic open subset of $\hat{A}$ of βD for which $p \in \hat{A} \subseteq U$

then $A \neq \phi$ {by def of $\hat{A} = \{p \in \beta D : A \in p\}$

لأنه $e(a)$ عنصر في $\hat{A}$

any $a \in A$ satisfy $e(a) = a \in D \cap \hat{A}$

So $D \cap \hat{A} \neq \phi$

So D has its limit point

(d) given any compact space Y and any function $f : D \longrightarrow Y$ there exists a cont func $g : \beta D \longrightarrow Y$ s.t $g|D = f$

proof/ Let $p \in \beta D$ and let U be a neigh of $g(p)$ in Y

Since Y is T_2 + compact $\Rightarrow Y$ is normal $\Rightarrow Y$ is regular

So we can find a neigh. V of $g(p)$ with $cl_Y V \subseteq U$ حسب lemma 31.1 p. 196 في منكر

let $A = f^{-1}[V]$

claim $A \in p$

suppose $D \setminus A \in p$

$\Rightarrow p \in \widehat{D \setminus A} = cl(D \setminus A)$

$\Rightarrow g(p) \in \overline{f(D \setminus A)}$

and since V is a neigh of $g(p)$

$\Rightarrow V \cap f[D \setminus A] \neq \phi$ $\times$ since $A = f^{-1}(V)$

So $A \in p$

$\Rightarrow p \in \hat{A} = \bar{A}$ is a neigh of p